MATH2050C Selected Solutions to Assignment 2

Section 2.2 no. 6, 7, 10, 11, 12, 14, 15.

Section 2.3 no. 3, 4, 5, 6, 7.

Supplementary Exercises

1. (a) Show that $(\mathbb{Z}_5, +, \cdot)$ (see Ex 1 for definition) does not have an order like $\mathbb{Q}$ or $\mathbb{R}$.

   (b) Explain why $\mathbb{Q}$ does not have the order-completeness property. How about $\mathbb{Q}(\sqrt{2})$?

   Solution. (a) Suppose there is an ordering. Then $1 = 1^2 \in \mathbb{P}$ but then $1+1+1+1+1+1 = 0(mod\ 5)$ shows that $0 \in \mathbb{P}$, impossible.

   (b) Consider the set $S = \{x \in \mathbb{Q} : x^2 < 2\}$. We claim that 3 is an upper bound for S: $x \in S \Rightarrow x^2 < 2 \Rightarrow x^2 < 9 \Rightarrow 9 - x^2 > 0 \Rightarrow (3+x)(3-x) > 0 \Rightarrow 3 - x > 0 \Rightarrow x < 3$, so S is bounded by 3 from above. Now, if the Order-Completeness Property holds for $\mathbb{Q}$, the supremum $u = \sup S$ would belong to $\mathbb{Q}$. However, we have already shown that there is no rational number whose square is 2. Therefore, the Order-Completeness Property does not hold for $\mathbb{Q}$.

2. Given $\varepsilon > 0$, show that there is some natural number n satisfying

$$\frac{1}{n} + \frac{a}{n^2} + \frac{b}{n^2} < \varepsilon,$$

where a, b are any real numbers.

Solution. Taking $\varepsilon_1 = \varepsilon/(1 + |a| + |b|)$, and applying Archimedean Property to ε_1, there is some natural number n such that

$$\frac{1}{n} < \varepsilon_1 \ .$$

Using this in the following

$$\begin{aligned} \frac{1}{n} + \frac{a}{n^2} + \frac{b}{n^2} &= \frac{1}{n}\left(1 + \frac{a}{n} + \frac{b}{n^2}\right) \\ &\le \frac{1}{n}\left(1 + \frac{|a|}{n} + \frac{|b|}{n^2}\right) \\ &\le \frac{1}{n}(1 + |a| + |b|) \\ &< \varepsilon \ . \end{aligned}$$